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Antiproximal Sets*

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The Banach space c_0 equipped with Day's norm is shown to contain an isomorph of the unit ball of c_0 (with the original norm) having the property that no point of its complement has a nearest point in it.

1. INTRODUCTION

A set S in a Banach space X is said to be antiproximal if no point in $X \setminus S$ has a nearest point in S. Let C be a closed and bounded symmetric convex body in X. Then, as can be readily seen, C is antiproximal if and only if the closed unit ball U is antiproximal in the norm induced by C. Thus antiproximality (for closed and bounded symmetric convex bodies) is symmetric in the pair U, C. Somewhat pictorially we call such pairs companion bodies. The existence of such bodies (in c_0) was the main theme of [2]. In it an isomorphism of c_0 onto itself was constructed such that U and its isomorph constitute a pair of companion bodies. The question raised there, whether c_0 equipped with Day's norm also possesses the same property, was answered in the affirmative, first by Cobzas [1] and later, independently, by R. C. O'Brien (private communication). Both authors employ the isomorphism of [2] to show that D, the closed unit ball in Day's norm, and its isomorph are companion bodies. It is the purpose of this note to show that D can also be matched with an isomorph of U to form a pair with the said property.

2. DAY'S NORM

We recall that Day's norm in c_0 is defined by

$$p(x) = \left(\sum_{j=1}^{\infty} (2^{-j}(x_{i_j}))^2 \right)^{1/2},$$

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where $(x_{i_1}, x_{i_2}, ..., x_{i_j}, ...)$ is a rearrangement of $(x_1, x_2, ...) = x \in c_0$ in such a manner that

$$|x_{i_1}| \geqslant |x_{i_2}| \geqslant \cdots . \tag{1}$$

This norm was shown by Rainwater [3] to be locally uniformly convex.

3. The Isomorphism A

Let $g_i \in l_1$, $i = 1, 2, ...$ be defined by setting $g_i = (a_{i1}, a_{i2}, ...)$, where $a_{ii} = 1$, $a_{i, 2^{i-1}(2n+1)} = 2^{-(n+1)}$, $n = 1, 2, ...$, and $a_{ij} = 0$ otherwise. In [2] it was pointed out that the linear operator $A: c_0 \to c_0$ defined by $(Ax)_i = g_i(x)$, $i = 1, 2, ...$ is bounded and has a bounded inverse. In addition it was observed that for any $g \in l_1$, $g \neq 0$, to attain its supremum on $A[U]$, g must be a nonzero finite linear combination of the functionals g_i. This last fact is especially useful since companion bodies are also characterized by the property that continuous linear functionals attain their suprema on at most one of them (cf. [2]).

Theorem 1. *Let D be the closed unit ball of c_0 in Day's norm (i.e., $D = \{x \in c_0 : p(x) \leqslant 1\}$) and let U be the closed unit ball of c_0 (in the usual norm). Then $A[U]$ is antiproximal.*

Proof. Let $g = (\lambda_1, \lambda_2, ...) \in l_1$, $g \neq 0$, be such that

$$g(x) = \sup\{g(z) : z \in A[U]\}.$$

We have to show that g fails to attain its supremum on D. As remarked before, g must be a nonzero linear combination of the functionals g_i defined there; i.e., $g = \sum_{i=1}^{m} \alpha_i g_i$ for some positive integer m and at least one $\alpha_i \neq 0$. Suppose then that a $y = (y_1, y_2, ...) \in D$ exists such that

$$g(y) = \sup\{g(z) : z \in D\}.$$

We may, and shall, assume that $a_i \geqslant 0$ $(i = 1, 2, ..., m)$, and $y_i \geqslant 0$ $(i = 1, 2, ...)$. Repeatedly, we shall make use of the elementary fact that, for any $\bar{y}$ in the interior of D,

$$g(\bar{y}) < g(y). \tag{2}$$

We distinguish between the two mutually exclusive possibilities:

(a) There exists a positive integer k_0 such that $y_{k_0} > 0$ and $y_k = 0$ for all $k > k_0$.

(b) There is no such k_0.

In case (a), $\lambda_{k_0} > 0$, as otherwise replacing y_{k_0} with zero produces a point $\bar{y}$ in the interior of D with $g(\bar{y}) = g(y)$, a contradiction of (2). Hence an i $(1 \leqslant i \leqslant m)$ must exist such that $a_i > 0$ and either $i = k_0$ or else $2^i(k_1 - 1) = k_0$ for some integer k_1. Let $\bar{y} = (\bar{y}_1, \bar{y}_2, ...)$ be defined by setting $\bar{y}_{k_0} = y_{k_0} - \varepsilon$, $\bar{y}_{2^i(2k_2-1)} = 2^{k_2}\lambda_{k_0}\varepsilon$, and $\bar{y}_l = y_l$ otherwise. Here $k_2 > k_0$, $0 < \varepsilon < y_{k_0}(1 + 2^{2k_2}\lambda_{k_0}^2)^{-1}$, and, moreover, ε is sufficiently small so that the rearrangement (1) of y is left intact.

We now have

$$g(\bar{y}) - g(y) = a_i(-\lambda_{k_0}\varepsilon + 2^{-k_2}2^{k_2}\lambda_{k_0}\varepsilon) = 0.$$

On the other hand,

$$\begin{aligned}(p(\bar{y}))^2 - (p(y))^2 &\leqslant \frac{(y_{k_0} - \varepsilon)^2 - y_{k_0}^2}{2^{2r}} + \frac{2^{2k_2}\lambda_{k_0}^2\varepsilon^2}{2^{2r}} \\ &= 2^{-2r}\varepsilon[-2y_{k_0} + \varepsilon(1 + 2^{2k_2}\lambda_{k_0}^2)] < 0.\end{aligned}$$

Here r is the index i_j corresponding to y_{k_0} in the rearrangement (1) of $\bar{y}$. Thus (2) applies, ruling out possibility (a).

In case (b) there exists a positive integer $k_0 \geqslant m + 1$ such that $y_{k_0} > 0$ and $y_k < y_{k_0}$ for all $k > k_0$. Clearly $k_0 = 2^{i_0}(2j_0 - 1)$ for some i_0 $(0 \leqslant i_0 \leqslant m)$ with $a_{i_0} > 0$ and some positive integer j_0. (If not then $\lambda_{k_0} = 0$ so that, for $\bar{y}$ obtained from y by replacing y_{k_0} with zero, $p(\bar{y}) < p(y)$ but $g(\bar{y}) = g(y)$, in violation of (2).) If now, $k_1 = 2^{i_0}(2j_0 + 1)$ then $0 < y_{k_1} < y_{k_0}$. (If not then, for $\bar{y}$ obtained from y by permuting y_{k_0} with y_{k_1}, $p(\bar{y}) = p(y)$ while $g(\bar{y}) > g(y)$, which is clearly impossible.) Let $0 < \varepsilon_1 < y_{k_0} - y_{k_1}$ and define $\bar{y} = (\bar{y}_1, \bar{y}_2, ...)$ by setting $\bar{y}_{k_0} = y_{k_0} - \varepsilon$, $\bar{y}_{k_1} = y_{k_1 + 2\varepsilon}$, and $\bar{y}_l = y_l$ otherwise. Here, too, $0 < \varepsilon < \varepsilon_1$ is small enough to leave the rearrangement (1) of y intact. Now,

$$g(\bar{y}) - g(y) = a_{i_0}(-\varepsilon\lambda_{k_0} + 2\varepsilon\lambda_{k_1}) = a_{i_0}(-\varepsilon 2^{-k_0} + 2\varepsilon 2^{-k_1}) = 0.$$

On the other hand, with $r = i_j$, where y_{i_j} corresponds to y_{k_0} in (1), we have

$$\begin{aligned}(p(\bar{y}))^2 - (p(y))^2 &\leqslant \frac{-2\varepsilon y_{k_0} + \varepsilon^2}{2^{2r}} + \frac{4\varepsilon y_{k_1} + \varepsilon^2}{2^{2(r+1)}} \\ &< 2^{-2r}\varepsilon(-2y_{k_0} + y_{k_1} + 2\varepsilon) \leqslant -y_{k_1} < 0;\end{aligned}$$

and again $p(\bar{y}) < p(y)$, contradicting (2).

Thus both cases (a) and (b) are ruled out, showing that g cannot achieve its supremum on D and proving the assertion of the theorem.

4. A Restatement of Theorem 1

Applying A^{-1} to D and $A[U]$ we arrive at

COROLLARY. *In c_0 there exists a closed and bounded symmetric body which is locally uniformly convex and antiproximal.*

References

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